Theorem 1. *A non-negative $f(x)$ is integrable on $[a, b]$. Then, $f^2(x)$ is integrable on $[a, b]$.*

Proof. Since f is integrable, it is bounded. Hence, we have $0 \leq f(x) \leq M$ for some constant M.

In addition, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$|U_f(P) - L_f(P)| < \frac{\epsilon}{2M}$$

holds if $|P| < \delta$. Namely,

$$\frac{\epsilon}{2M} > U_f(P) - L_f(P) = \sum_{i=1}^{n} (x_i - x_{i-1})(M_i - m_i).$$

where $P = \{x_i\}_{i=0}^n$, $M_i = \sup_{[x_{i-1}, x_i]} f$, and $m_i = \inf_{[x_{i-1}, x_i]} f$.

Now, $f \geq 0$ implies

$$M_i^2 = \sup_{[x_{i-1}, x_i]} f^2, \qquad m_i^2 = \inf_{[x_{i-1}, x_i]} f^2.$$

Moreover, $0 \leq f(x) \leq M$ yields $0 \leq m_i, M_i \leq M$. Hence,

$$\begin{aligned}
|U_{f^2}(P) - L_{f^2}(P)| &= \sum_{i=1}^{n} (x_i - x_{i-1})(M_i^2 - m_i^2) \\
&= \sum_{i=1}^{n} (x_i - x_{i-1})(M_i - m_i)(M_i + m_i) \\
&\leq \sum_{i=1}^{n} (x_i - x_{i-1})(M_i - m_i)(2M) \\
&= 2M|U_f(P) - L_f(P)| < \epsilon.
\end{aligned}$$

Thus, f^2 is integrable on $[a, b]$. □